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The Peierls–Nabarro and Benjamin–Ono Equations

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An intimate connection between the Peierls–Nabarro equation in crystal-dislocation theory and the travelling-wave form of the Benjamin–Ono equation in hydrodynamics is uncovered. It is used to prove the essential uniqueness of Peierls' solution of the Peierls–Nabarro equation and to give, in closed form, all solutions of the analogous periodic problem. The latter problem is shown to be an example of

1. INTRODUCTION

When constants have been normalised to equal 1, the travelling-wave form of the Benjamin–Ono equation in theoretical hydrodynamics [5] is the nonlinear pseudo-differential operator equation

$$\mathscr{H}(\phi') = \phi^2 - \phi \qquad \text{on } \mathbb{R} \tag{B–O}$$

where ϕ is a bounded function on $\mathbb{R}$, prime denotes differentiation, and $\mathscr{H}$, the Hilbert transform on $L_2(\mathbb{R})$, is given by the Cauchy principal value integral formula

$$\mathscr{H}(v)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(t)}{x-t}\, dt,$$

or equivalently by $(\mathscr{H}v)^\wedge(k) = i(\operatorname{sign} k)\, v^\wedge(k)$, where $v^\wedge$ is the Fourier transform of $v \in L_2(\mathbb{R})$. See [12].

Once constants have been normalised similarly, the Peierls–Nabarro equation, which arises in continuum modelling of dislocations in crystals [9], is

$$\mathscr{H}(\psi') = \sin \psi \qquad \text{on } \mathbb{R}, \tag{P–N}$$

which is to be satisfied by a bounded function ψ.

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It is known [1, 2] that, apart from constants, all bounded solutions of (B–O) are translates of the solitary-wave solution ϕ_*, which Benjamin [5] found; here

$$\phi_*(x) = \frac{2}{1+x^2}, \qquad x \in \mathbb{R}. \tag{1.1}$$

We will use this fact to prove that the only non-constant bounded solutions of (P–N) are of the form

$$\psi(x) = \pm \psi_*(x+a) + 2\pi n, \tag{1.2}$$

where $\psi_*(x) = 2\tan^{-1}(x)$ and $a \in \mathbb{R}$ and $n \in \mathbb{N}$ are arbitrary. This establishes the essential uniqueness of the solution ψ_* of (P–N) discovered by Peierls [10] (see also [9]). Even though the symmetry group for (P–N) is much larger than that for (B–O), the observation that $\phi_* = d\psi^*/dx$ is not a coincidence. An explanation is given in Theorem 1, which says among other things that the derivative of any solution of (P–N) is the difference of two solutions of (B–O). Note that changing the sign of the sine function in (P–N) is equivalent to leaving the sign unchanged and adding π to the solution instead. Since all solutions are known in closed form, the equation with the opposite sign for the sine function may be treated by changing variables.

2. MORE GENERALITY

For any function $v \in V = \bigcup_{1 \leqslant p < \infty} L_p(\mathbb{R})$, the Hilbert transform is defined for almost all x by

$$\mathscr{H} v(x) = \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_{|x-t| > \varepsilon} \frac{v(t)}{x-t}\, dt. \tag{2.1}$$

Hence $\mathscr{H} w$ is defined almost everywhere when w is a finite sum of elements from V. Let $X = \sum_{1 \leqslant p < \infty} L_p(\mathbb{R})$ denote the space of all such finite sums. Now (B–O) and (P–N) are special cases of the equation

$$\mathscr{H}(u') = F(u) \tag{2.2}$$

for a (locally) absolutely continuous function u with $u' \in X$. Here F is smooth. From [14, Theorem 9], it follows that $F(u)$ and u' are in $\sum_{1 < p < \infty} L_p(\mathbb{R})$ and $u \in L_\infty(\mathbb{R})$ if u satisfies (2.2) when $F(u) = \sin u$ or $F(u) = u^2 - u$. Since $u \in L_\infty(\mathbb{R})$ and $u' = -\mathscr{H}(F(u))$ it follows that $u'' \in \sum_{1 < p < \infty} L_p(\mathbb{R})$. A bootstrap argument now yields that all the derivatives of u are in $\sum_{1<p<\infty} L_p(\mathbb{R})$ and hence u is a smooth, bounded function. (Note that u itself may not be in $L_p(\mathbb{R})$ for any $p<\infty$.) Suppose u satisfies (2.2) and let

$$U(x, y) = \begin{cases} u(x) & \text{if} \quad y=0 \\ \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, u(t)\, dt & \text{if} \quad y>0. \end{cases}$$

Then (see [3, Chap. 7], [7, Theorem 6.4] or [12, p. 65]) u is a bounded harmonic function on the open upper half-plane Ω which is continuously differentiable on its closure. Indeed, by the Phragmèn–Lindelöf Principle (see [3, 6, 11]),

$$\sup_{x\in\mathbb{R},\, y\geqslant 0} |U(x, y)| = \sup_{x\in\mathbb{R}} |u(x)|,$$

and on the boundary the Cauchy–Riemann equations give

$$\frac{\partial U}{\partial y}(x, 0) + F(U(x, 0)) = 0.$$

Hence, the existence of a (locally) absolutely continuous function ψ, with $\psi' \in X$, satisfying (P–N) is a special case of the following nonlinear Neumann boundary-value problem:

$$\left.\begin{aligned} & f\in C^\infty(\Omega)\cap C^1(\bar{\Omega}); \\ & \Delta f(x, y) = 0, \qquad x\in\mathbb{R},\ y>0; \\ & f \text{ is bounded on } \Omega; \\ & \frac{\partial f}{\partial y} + \sin f = 0 \text{ on } y=0. \end{aligned}\right\} \tag{2.3}$$

Similarly the existence of a (locally) absolutely continuous function ϕ, with $\phi' \in X$, satisfying (B–O) is a special case of the problem

$$\left.\begin{aligned} & u\in C^\infty(\Omega)\cap C^1(\bar{\Omega}); \\ & \Delta u(x, y) = 0, \qquad x\in\mathbb{R},\ y>0; \\ & u \text{ is bounded on } \Omega; \\ & \frac{\partial u}{\partial y} - u + u^2 = 0 \text{ on } y=0. \end{aligned}\right\} \tag{2.4}$$

It follows from Lewy's theorem [8] that the solutions f and u have harmonic extensions to an open set which contains the upper half-plane, and both are therefore real-analytic on $\bar{\Omega}$. Note that no assumption is made about the behaviour of f or u at infinity apart from boundedness and whatever is implicit in the fact that $\phi', \psi' \in X$. Therefore the characterization of all solutions of (P–N) given in Section 5 involves no such assumption either. Our main result is the following:

THEOREM 1. *If f is a solution of* (2.3), *then there exist two solutions*, v *and* u, *of* (2.4) *such that*

$$f_x(x, y) = v(x, y) - u(x, y), \qquad x \in \mathbb{R}, \quad y > 0,$$

$$f_x(x, y)\, f_y(x, y) = v_x(x, y) + u_x(x, y), \qquad x \in \mathbb{R}, \quad y > 0,$$

and

$$1 + \cos f(x, 0) = v(x, 0) + u(x, 0), \qquad x \in \mathbb{R}.$$

Remark. It is easy to see that

$$f(x, y) = 2 \tan^{-1}\left(\frac{x}{1+y}\right), \qquad x \in \mathbb{R}, \quad y > 0,$$

gives a solution of (2.3) which coincides with ψ_*, Peierls' solution of (P–N), on the real axis. (It may be interesting to note that this solution f of (2.3) is a harmonic conjugate of $\log(x^2 + (1+y)^2)$, the fundamental solution of the Laplacian on $\mathbb{R}^2$.) In this case the solutions of (2.4) in Theorem 1 are

$$v(x, y) = \left(\frac{2(1+y)}{x^2 + (1+y)^2}\right) \qquad \text{and} \qquad u(x, y) \equiv 0.$$

Note that $v = \phi_*$ on $y = 0$. If f is replaced by $-f$, which is also a solution of (2.3), then the roles of u and v are swapped. *All* solutions of (2.4) are known in closed form and it is the work of Section 5 to deduce from Theorem 1 that the only solutions of (2.3) which are not periodic in x are those of the form

$$f(x, y) = \pm 2 \tan^{-1}\left(\frac{x+a}{1+y}\right) + 2\pi n, \qquad x \in \mathbb{R}, \quad y > 0,$$

where $a \in \mathbb{R}$ and $n \in \mathbb{N}$ are arbitrary. (This shows that Peierls' solution of (P–N) is essentially unique.) All solutions of (2.4), apart from translates of Benjamin's solitary-wave solution and the constants 0 and 1, are periodic and are described as follows [1; 2, Eq. (1.14)]. For each $\alpha \in (1, 2)$, let

$$\beta = \alpha^2 - 2\alpha, \qquad \delta = \tfrac{1}{2}\sqrt{-\beta}, \qquad \gamma = \alpha/\sqrt{-\beta} \tag{2.5}$$

and

$$\Gamma_\alpha(y) = \left(\frac{\gamma + \tanh(\delta y)}{1 + \gamma \tanh(\delta y)}\right), \qquad y \geqslant 0.$$

Then

$$u_\alpha(x, y) = \frac{2\delta\Gamma_\alpha(y)}{\cos^2(\delta x) + \Gamma_\alpha(y)^2 \sin^2(\delta x)} \tag{2.6}$$

is (π/δ)-periodic in x and is a positive solution of (2.4) with $u(0) = \alpha = u_{\max}$. (With $\alpha \in (0, 1)$, these formulae correspond to a different parametrisation of the same family of periodic solutions; now $\alpha = u(0) = u_{\min}$.)

In Section 5 it is proved that if f is any solution of (2.3) then, in Theorem 1, v must be a reflection of u about a point half-way between consecutive maximisers and minimizers of u. This leads to a proof that all non-constant solutions of (2.3), apart from Peierls', are of the form

$$f(x, y) = \pm f_\alpha(x + a, y) + 2\pi n, \qquad a \in \mathbb{R}, \quad n \in \mathbb{N},$$

where

$$f_\alpha(x, y) = 2\left\{\tan^{-1}\left(\frac{\tan \delta x}{\Gamma_\alpha(y)}\right) - \tan^{-1}(\Gamma_\alpha(y) \tan \delta x)\right\}, \qquad \alpha \in (1, 2).$$

Moreover, $u_\alpha(x, y) \to 1$ uniformly as $\alpha \to 1$, $u_\alpha(x, y) \to 2(1 + y)/\{x^2 + (1 + y)^2\}$, uniformly on compact sets as $\alpha \to 2$. Note that $f_\alpha(x, 0) \to \psi_*(x)$, uniformly on compact intervals of $\mathbb{R}$ as $\alpha \to 2$, and $f_\alpha(x, 0) \to 0$ uniformly on $\mathbb{R}$ as $\alpha \to 1$.

3. PERIODIC SOLUTIONS, THE CONJUGATE OPERATOR, AND GLOBAL BIFURCATION THEORY

If a solution f of (2.3) is periodic in x of period λ, $\lambda > 0$, let $\mu = \lambda/2\pi$ and

$$g(x, y) = f(\lambda x/2\pi, \lambda y/2\pi).$$

Then

$$\begin{aligned}
&\Delta g(x, y) = 0, && x \in \mathbb{R}, \quad y > 0,\\
&|g(x, y)| \leqslant M, && x \in \mathbb{R}, \quad y > 0,\\
&g(x, y) = g(x + 2\pi, y), && x \in \mathbb{R}, \quad y > 0,\\
&g_y + \mu \sin g = 0, && x \in \mathbb{R}, \quad y = 0.
\end{aligned}$$

This is a standard bifurcation problem (once mapped conformally into the unit disc it is a nonlinear Steklov problem in the sense of [13] where the trivial solution is $g=0$ for all $\mu \in \mathbb{R}$. The bifurcation points are $\mu_n = n$, $n \in \mathbb{N}$. The preceding uniqueness result shows that, modulo translations, there is a branch of solutions bifurcating from μ_n for each n, and that each branch may be obtained from the first one by rescaling. Further, there is no secondary bifurcation on any of these branches. Finally, the convergence of f_α to ψ_* as $\alpha \to 2$, yields the asymptotic behaviour of solutions on the branches as $\mu \to \infty$.

This periodic problem may be written as

$$\mathscr{C}(g') = \mu \sin g,$$

where $\mathscr{C}$ is the conjugate operator in the theory of Fourier series defined by the Cauchy principal value integral

$$\mathscr{C}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v(x-t)}{\tan(t/2)}\, dt$$

for any 2π-periodic function v with $v \in L_p(-\pi, \pi)$, $1 \leqslant p \leqslant \infty$ (see [4, 15]).

4. PROOF OF THEOREM 1

Suppose f is a solution of (2.3). Then f is smooth and each of its partial x-derivatives is a bounded function on the line $y=0$. The classical gradient estimate for a harmonic function f ([7], Theorem 2.10) now gives

$$|\nabla f(x, y)| \leqslant \text{const.}(1+y)^{-1}. \tag{4.1}$$

Let g be a harmonic conjugate of f so that $f + ig$ is analytic on the upper half-plane. Then the Cauchy–Riemann equations

$$f_x(x, y) = g_y(x, y), \qquad f_y(x, y) = -g_x(x, y) \tag{4.2}$$

are satisfied. Now let

$$w(x, y) = f_{xx}(x, y) + f_x(x, y)\, g_x(x, y). \tag{4.3}$$

Then w is harmonic on the upper half-plane and by (4.2)

$$\begin{aligned} w(x, 0) &= f_{xx}(x, 0) - f_x(x, 0)\, f_y(x, 0) \\ &= f_{xx}(x, 0) + f_x(x, 0) \sin f(x, 0) \\ &= \frac{\partial}{\partial x} \{ f_x(x, 0) - \cos f(x, 0) \}. \end{aligned} \tag{4.4}$$

Also

$$\begin{aligned}
w_y(x,0) &= \{f_{xxy}+f_{xy}g_x+f_xg_{xy}\}|_{(x,0)}\\
&= \{(-\sin f)_{xx}-(\sin f)_x \sin f+f_x f_{xx}\}|_{(x,0)}\\
&= \{-f_{xx}\cos f+f_x^2\sin f-f_x\sin f\cos f+f_x f_{xx}\}|_{(x,0)}\\
&= \{f_{xx}(f_x-\cos f)+f_x\sin f(f_x-\cos f)\}|_{(x,0)}\\
&= \{(f_x-\cos f)\, w\}|_{(x,0)}\\
&= \frac{1}{2}\frac{\partial}{\partial x}\{(f_x-\cos f)^2|_{(x,0)}\},
\end{aligned} \tag{4.5}$$

by (4.4).

For any $x \in \mathbb{R}$, $y > 0$, let

$$W(x,y) = -f_x(0,y) - \frac{1}{2}\int_y^\infty [f_y^2(0,s)-f_x^2(0,s)]\,ds - \int_0^x w(t,y)\,dt. \tag{4.6}$$

By (4.1),

$$|W(x,y)| \leqslant \text{const.} + \left|\int_0^x w(t,y)\,dt\right|.$$

Also, by (4.1) and (4.3),

$$\left|\int_0^x w(t,y)\,dt\right| \leqslant \text{const.} + \left|\int_0^x f_x(t,y)\,g_x(t,y)\,dt\right|.$$

However, from Cauchy's integral formula applied to the analytic function $(f_x+ig_x)^2$ on the rectangle with vertices $(0,0)$, $(x,0)$, (x,y), $(0,y)$, we find that

$$\begin{aligned}
2\left|\int_0^x f_x(t,y)\,g_x(t,y)\,dt\right| \leqslant{}& 2\left|\int_0^x f_x(t,0)\,g_x(t,0)\,dt\right|\\
&+\int_0^y |f_x(0,s)|^2+|g_x(0,s)|^2\,ds\\
&+\int_0^y |f_x(x,s)|^2+|g_x(x,s)|^2\,ds.
\end{aligned}$$

The first term on the right-hand side is bounded by 4, because of (2.3) and (4.2). The other two terms are bounded independent of (x, y) in the upper half-plane, because of (4.1) and (4.2). Therefore $|W(x, y)|$ is bounded in the upper half-plane. Finally, note from (4.6) that

$$W_{xx}(x, y) = -w_x(x, y) = -w_x(0, y) - \int_0^x w_{xx}(t, y)\, dt$$

and

$$W_{yy}(x, y) = \{-f_{xyy} + f_y f_{yy} - f_x f_{xy}\}|_{(0, y)} - \int_0^x w_{yy}(t, y)\, dt.$$

But w is harmonic and

$$-w_x = -f_{xxx} - f_{xx} g_x - f_x g_{xx} = f_{xyy} - f_y f_{yy} + f_x f_{xy},$$

since f is harmonic and the Cauchy–Riemann equations hold. Therefore W is a bounded harmonic function on the upper half-plane.

Moreover, by (4.4) and (4.6),

$$\begin{aligned} W(x, 0) &= -f_x(0, 0) - \tfrac{1}{2} \int_0^\infty [f_y^2(0, s) - f_x^2(0, s)]\, ds - \int_0^x w(t, 0)\, dt \\ &= A - f_x(x, 0) + \cos f(x, 0), \end{aligned} \tag{4.7}$$

where

$$A = -\tfrac{1}{2} \int_0^\infty (f_y^2(0, s) - f_x^2(0, s))\, ds - \cos f(0, 0).$$

Now, by (2.3) and (4.5),

$$\begin{aligned} W_y(x, 0) &= -f_{xy}(0, 0) + \tfrac{1}{2} f_y^2(0, 0) - \tfrac{1}{2} f_x^2(0, 0) - \int_0^x w_y(t, 0)\, dt \\ &= \{f_x \cos f + \tfrac{1}{2} \sin^2 f - \tfrac{1}{2} f_x^2 + \tfrac{1}{2} (f_x - \cos f)^2\}|_{(0, 0)} \\ &\quad - \tfrac{1}{2} (f_x(x, 0) - \cos f(x, 0))^2 \\ &= \tfrac{1}{2} [1 - (f_x(x, 0) - \cos f(x, 0))^2] \\ &= \tfrac{1}{2} (1 - A + W(x, 0))(1 + A - W(x, 0)). \end{aligned} \tag{4.8}$$

If

$$u(x, y) = \tfrac{1}{2}(1 - A + W(x, y)), \qquad x \in \mathbb{R}, \quad y > 0, \tag{4.9}$$

then u is bounded and harmonic on the upper half-plane and by (4.8)

$$u_y(x, 0) = u(x, 0) - u^2(x, 0).$$

Therefore u is a solution of (2.4) and by (4.7)

$$u(x, 0) = \tfrac{1}{2}\{1 - f_x(x, 0) + \cos f(x, 0)\}. \tag{4.10}$$

Finally, note that if f is a solution of (2.3) then $-f$ is also a solution of (2.3). Hence there exists a solution v of (2.4) such that

$$v(x, 0) = \tfrac{1}{2}\{1 + f_x(x, 0) + \cos f(x, 0)\}. \tag{4.11}$$

Thus, adding (4.10) and (4.11) gives

$$1 + \cos f(x, 0) = u(x, 0) + v(x, 0), \tag{4.12}$$

and subtracting them gives

$$f_x(x, 0) = v(x, 0) - u(x, 0). \tag{4.13}$$

But f_x and $v - u$ are bounded harmonic functions on the upper half-plane which coincide on the real axis. Hence, they are equal everywhere by the Phragmèn–Lindelöf principle [3, 6, 11]. Differentiation of (4.9) gives

$$2u_x(x, y) + f_{xx}(x, y) + f_x(x, y)\, g_x(x, y) = 0.$$

Similarly it follows that

$$2v_x(x, y) - f_{xx}(x, y) + f_x(x, y)\, g_x(x, y) = 0.$$

Adding these gives

$$u_x(x, y) + v_x(x, y) + f_x(x, y)\, g_x(x, y) = 0.$$

Therefore

$$u_x(x, y) + v_x(x, y) = f_x(x, y)\, f_y(x, y),$$

by the Cauchy–Riemann equations. This completes the proof.

5. RAMIFICATIONS OF THEOREM 1

To exploit Theorem 1, it is useful to list some properties of solutions of (2.4) which were established by Amick and Toland [1] using the maximum principle and complex ordinary differential equations.

Any non-constant solution u of (2.4) has the following properties:

$$\text{if } u \not\equiv 0 \text{ then } u > 0 \text{ everywhere;} \tag{5.1}$$

there exists $\beta \leqslant 0$ such that

$$u_x^2(x, 0) = \beta u^2(x, 0) + 2u^3(x, 0) - u^4(x, 0), \qquad x \in \mathbb{R}, \tag{5.2}$$

$$\sup_{x \in \mathbb{R}} u(x, 0) + \inf_{x \in \mathbb{R}} u(x, 0) = 2. \tag{5.3}$$

When $u = u_\alpha$, defined in (2.6), then $u(0, 0) = \alpha$ and β in (5.2) is the same as β in (2.5). Note also that (5.3) is an easy consequence of (5.2) in this case. When $\beta = 0$ in (5.2), $\alpha = 2$ and

$$u_2(x, y) = \frac{2(1 + y)}{(1 + y)^2 + x^2}, \qquad x \in \mathbb{R}, \quad y > 0. \tag{5.4}$$

Let

$$u_0(x, y) = 0, \qquad u_1(x, y) = 1, \qquad x \in \mathbb{R}, \quad y > 0. \tag{5.5}$$

Then u_α, $\alpha \in \{0\} \cup [1, 2]$ are, except for translates, the only solutions of (2.4).

Now suppose that f is a solution of (2.3) and, as in Theorem 1,

$$f_x(x, y) = v(x, y) - u(x, y), \tag{5.6}$$

$$1 + \cos f(x, 0) = v(x, 0) + u(x, 0). \tag{5.7}$$

When it is convenient, we will write $u(x)$ instead of $u(x, 0)$, and similarly for v and f.

We begin by dealing with the case where u and v are both constant functions, 0 or 1. Because of (5.6) and since f is bounded, the only possibilities are $u \equiv v \equiv 1$ or $u \equiv v \equiv 0$, in this case. Because of (5.7), for some $n \in \mathbb{Z}$

$$f = 2n\pi \qquad \text{on } \mathbb{R} \qquad \text{if} \quad u \equiv v \equiv 1$$

and

$$f = (2n + 1)\,\pi \qquad \text{on } \mathbb{R} \qquad \text{if} \quad u \equiv v \equiv 0.$$

Since f is a bounded harmonic function in the upper half-plane, f is constant everywhere, by the Phragmèn–Lindelöf principle.

Next note that if one, but not both of u and v is zero, then the boundedness of f, together with (5.1) and (5.6), means that the non-constant u or v must be integrable on $\mathbb{R}\times\{0\}$. Hence since all the possibilities are offered by (2.6), (5.4), and (5.5), the non-constant solution of (2.4) must be given by (5.4) or a translate of it in the x-direction. It follows by (5.6) and (5.7) that f on $\mathbb{R}\times\{0\}$ is one of the functions

$$2n\pi \pm 2\tan^{-1}(x+a) \qquad \text{for some} \quad a\in\mathbb{R}, \quad n\in\mathbb{Z}.$$

To deal with the case where one but not both of u and v is 1, note from the boundedness of f and (5.6) that the other of u and v cannot be integrable. The only possibility is that it is u_α for some $\alpha\in(1,2)$. But u_α in (2.6) has period π/δ and

$$\int_0^{\pi/\delta} u_\alpha(x,y)\,dx = 2\pi,$$

independent of α and $y>0$. Since f is bounded, it follows from (5.6) that $\pi/\delta = 2\pi$, i.e., $\alpha=1$, which corresponds to the constant function 1. Hence if $u\equiv 1$ then $v\equiv 1$ and *vice versa*.

Next suppose that u is given by (5.4) (or a translate). Then $\sup_{x\in\mathbb{R}} u(x) = 2$ and it is attained. Hence, since $|\cos f|\leqslant 1$, it follows from (5.7) and (5.1) that $v\equiv 0$, which is a preceding case. A similar result holds if v is given by (5.4).

The only remaining case is the one where neither u nor v is a constant and neither is given by a translate of (5.4). In other words, both u and v are given by translates of (2.6), with possibly different values of α. Note that both u and v then satisfy the differential equation (5.2), but with possibly different values of β.

Suppose (seeking a contradiction) that $f_x>0$ on $\mathbb{R}$. Then $v>u$ on $\mathbb{R}$ by (5.6). If $\hat{x}$ is a point of $\mathbb{R}$ where $\sup_{x\in\mathbb{R}} u$ is attained then, by (5.7),

$$\begin{aligned} 1+\cos f(\hat{x}) &= v(\hat{x}) + u(\hat{x}) \\ &> 2u(\hat{x}) > \{\sup_{x\in\mathbb{R}} u(x) + \inf_{x\in\mathbb{R}} u(x)\} \\ &= 2, \qquad \text{by (5.3).} \end{aligned}$$

This is false because $\cos f(\hat{x})\leqslant 1$. Hence $f_x \not> 0$ on $\mathbb{R}$. Similarly it follows that $f_x \not< 0$ on $\mathbb{R}$. Hence there exists $x_0\in\mathbb{R}$ such that $f_x(x_0)=0$. From (5.6) and (5.7) we now see that

$$v(x_0) = u(x_0) = 0 = v_x(x_0) + u_x(x_0). \tag{5.8}$$

Since $v(x_0)=u(x_0)$ and $u(x_0)\neq 0$ because of (5.1), it follows that u and v satisfy equation (5.2) for the same value of β, β_0 say, where $\beta_0<0$. Therefore, by the uniqueness theorem for initial-value problems, from (5.2) and (5.8) we conclude that

$$v(x_0+x)=u(x_0-x) \qquad \text{for all} \quad x\in\mathbb{R}. \tag{5.9}$$

Note that $u_x(x_0)\neq 0$, for if it were then $u(x_0+x)=u(x_0-x)=v(x_0+x)$ and hence $u\equiv v$ on $\mathbb{R}$, from which it follows, by the Phragmèn–Lindelöf principle that $v=u$ on the upper half-plane. Then, by (5.6) $f=$constant, and $v=u=$constant on $\mathbb{R}$, by (5.7). Again $v=u$ on the upper half-plane. This case has been dealt with earlier. Therefore x_0 is neither a maximiser nor a minimiser of u. Since u is one of the functions given by (2.6), it has exactly one maximiser and one minimiser per period and it is even about each maximiser and each minimiser. (This can be seen from (5.2) without recourse to the formula for u.) Hence, since x_0 is neither a maximiser nor a minimiser of u, there is a unique point $\hat{x}$ closest to x_0 where u attains its maximum.

By (5.7) and (5.9)

$$\begin{aligned} 1+\cos f(x_0+x) &= v(x_0+x)+u(x_0+x) \\ &= u(x_0-x)+u(x_0+x), \qquad x\in\mathbb{R}, \end{aligned} \tag{5.10}$$

and therefore, by (5.3),

$$2\geqslant 1+\cos f(\hat{x})=u(2x_0-\hat{x})+u(\hat{x})\geqslant \inf_{x\in\mathbb{R}} u(x)+\sup_{x\in\mathbb{R}} u(x)=2. \tag{5.11}$$

Hence

$$u(2x_0-\hat{x})=\min_{x\in\mathbb{R}} u(x) \qquad \text{and} \qquad \cos f(\hat{x})=1. \tag{5.12}$$

Therefore $\hat{x}$ is a maximiser and $2x_0-\hat{x}$ is a minimiser of u and x_0 lies mid-way between these two. Since u has one maximiser and one minimiser per period and since $\hat{x}$ is the maximiser nearest to x_0, it follows that $2x_0-\hat{x}$ is the minimiser nearest to x_0. Hence consecutive maximisers and minimisers are separated by a distance $|2(x_0-\hat{x})|$. Since u is clearly (from (2.6) or (5.2)) even about maximisers and minimisers we conclude that u has period $4\,|x_0-\hat{x}|$.

Hence

$$4\,|x_0-\hat{x}|=\pi/\delta, \tag{5.13}$$

where δ depends on α according to (2.5) and $\alpha\in(1,2)$. Because of the limited number of possibilities offered by (2.6), there is no loss of essential generality in removing the transition invariance in the x-direction by supposing that

$$\hat{x}=0, \qquad x_0=\pi/4\,\delta \qquad \text{and} \qquad u=u_\alpha, \qquad \alpha\in(1,2), \tag{5.14}$$

where δ and u_α are given by (2.5) and (2.6). (The other possible choice, $x_0=3\pi/4\,\delta$, $\hat{x}=\pi/\delta$, results only in a change of sign of f_α in the subsequent analysis and leads to the $\pm$ in the formula for the general periodic solution of (2.3) given at the end of Section 2.) Then, by (5.6) it follows that

$$f_x(x_0+x,\,y)=u_\alpha(x_0-x,\,y)-u_\alpha(x_0+x,\,y), \tag{5.15}$$

from which it follows that both f and f_x are (π/δ)-periodic. By (5.12) and (5.14), $f(0,0)=0 \bmod (2\pi)$. It therefore follows by (5.14) and (5.15) that

$$f(x,0)=2n\pi+\int_0^x\left\{u_\alpha\left(\frac{\pi}{2\,\delta}-s,\,0\right)-u_\alpha(s,0)\right\}ds, \qquad x\in\mathbb{R},$$

for some $n\in\mathbb{Z}$. Now let

$$f_\alpha(x,\,y)=\int_0^x\left\{u_\alpha\left(\frac{\pi}{2\,\delta}-s,\,y\right)-u_\alpha(s,\,y)\right\}ds, \qquad x\in\mathbb{R}, \quad y>0. \tag{5.16}$$

Then f_α is bounded function on the upper half plane which is harmonic because $(u_\alpha)_x\,(\pi/2\,\delta,\,y)=(u_\alpha)_x\,(0,\,y)=0$ for all $y\geqslant 0$.

Since $f-2\pi n$ is a bounded harmonic function on the upper half-plane which coincides with f_α on $y=0$ it follows that

$$f(x,\,y)=2\pi n+f_\alpha(x,\,y), \qquad x\in\mathbb{R}, \quad y\geqslant 0. \tag{5.17}$$

Next we confirm that every function f given by (5.17) for some $\alpha\in(1,2)$ satisfies (2.3). We claim that to do this it suffices to verify that f satisfies (5.10); i.e.,

$$1+\cos f(x_0+x)=u_\alpha(x_0-x,0)+u_\alpha(x_0+x,0) \qquad \text{with} \quad x_0=\pi/4\,\delta. \tag{5.18}$$

To see that this is enough, suppose that (5.16) and (5.17) hold. Then (5.15) follows from (5.16). A differentiation of (5.15) with respect to y at $y=0$, along with the fact that u_α satisfies (2.4) and further substitutions from (5.15) and (5.18) yields the identity

$$f_{xy}(x,0)+f_x(x,0)\cos f(x,0)=0, \qquad x\in\mathbb{R}.$$

Since $f_y(0,0) = 0 = \sin f(0,0)$, by (5.17) and (5.16), it follows that

$$f_y(x,0) + \sin f(x,0) = 0, \qquad x \in \mathbb{R},$$

as required. It remains only to verify (5.10). By (5.17) it suffices to confirm that

$$\cos f_\alpha(x,0) = u_\alpha\left(\frac{\pi}{2\,\delta} - x, 0\right) + u_\alpha(x,0) - 1, \qquad x \in \mathbb{R}. \tag{5.19}$$

By (2.6), the right-hand side of (5.19) is equal to

$$\begin{aligned}
&2\,\delta\gamma\left\{\frac{1}{\sin^2(\delta x) + \gamma^2\cos^2(\delta x)} + \frac{1}{\cos^2(\delta x) + \gamma^2\sin^2(\delta x)}\right\} - 1\\
&\quad = 2\,\delta\gamma\left\{\frac{1+\gamma^2}{(1+(\gamma^2-1)\cos^2(\delta x))(1+(\gamma^2-1)\sin^2(\delta x))}\right\} - 1\\
&\quad = 2\,\delta\gamma\left\{\frac{1+\gamma^2}{\gamma^2 + \frac{1}{4}(\gamma^2-1)\sin^2(2\,\delta x)}\right\} - 1\\
&\quad = \left\{\frac{8\gamma^2}{4\gamma^2 + (\gamma^2-1)\sin^2(2\,\delta x)}\right\} - 1,
\end{aligned} \tag{5.20}$$

by (2.5). Now by (5.16) and (2.6)

$$\begin{aligned}
f_\alpha(x,0) &= \int_0^x \left\{\frac{2\,\delta\gamma}{\sin^2(\delta t) + \gamma^2\cos^2(\delta t)} - \frac{2\,\delta\gamma}{\cos^2(\delta t) + \gamma^2\sin^2(\delta t)}\right\} dt\\
&= \int_0^x \left\{\frac{2\,\delta\gamma\sec^2(\delta t)}{\tan^2(\delta t) + \gamma^2} - \frac{2\,\delta\gamma\sec^2(\delta t)}{1 + \gamma^2\tan^2(\delta t)}\right\} dt\\
&= 2\tan^{-1}\left(\frac{1}{\gamma}\tan(\delta x)\right) - 2\tan^{-1}(\gamma\tan(\delta x)).
\end{aligned}$$

Hence

$$\tan\left(\frac{1}{2} f_\alpha(x,0)\right) = \frac{1}{2}\left(\frac{1}{\gamma} - \gamma\right)\sin(2\,\delta x),$$

and therefore

$$\begin{aligned}
\cos\, f_\alpha(x,0) &= \frac{4\gamma^2 - (\gamma^2-1)^2\sin^2(2\,\delta x)}{4\gamma^2 + (\gamma^2-1)^2\sin^2(2\,\delta x)}\\
&= \frac{8\gamma^2}{4\gamma^2 + (\gamma^2-1)^2\sin^2(2\,\delta x)} - 1.
\end{aligned} \tag{5.21}$$

Together (5.20) and (5.21) ensure that (5.19) holds. Hence f defined by (5.17) is a periodic solution of (2.3). Once translations in the x-direction and a change in the sign of f_α (see the remark in parentheses following (5.14)) have been included we have found all the solutions of (2.3) in closed form. They are all translations in the x-direction of (i) the constants, $n\pi$, $n \in \mathbb{N}$, (ii) Peierls' solutions ψ_* and $2\pi n \pm \psi_*$, $n \in \mathbb{N}$ and (iii) the periodic solutions listed at the end of Section 2.

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